EPFL– Fall 2025
SOLUTIONS: Series 4

Differential Geometry IV: General Relativity

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4.1 Let $\mathcal{M}^n$ be a differentiable manifold.

(a) Show that, for any $X, Y, Z \in \Gamma(\mathcal{M})$:

$$\mathcal{L}_{[X,Y]}Z = \mathcal{L}_X\mathcal{L}_Y Z - \mathcal{L}_Y\mathcal{L}_X Z.$$

Show that the above relation also holds when Z is replaced by any tensor field f of type (k,l), $k,l \in \mathbb{N}$. *(Hint: Check how $\mathcal{L}_X$ behaves on tensor products of the form $f_1 \otimes f_2$.)*

(b) Show that the space $\mathcal{K}$ of Killing vector fields on $(\mathcal{M}, g)$ is closed under commutation, i.e. that $[X,Y] \in \mathcal{K}$ if $X, Y \in \mathcal{K}$; thus, $\mathcal{K}$ forms a Lie subalgebra of $\Gamma(\mathcal{M})$.

(c) Let g be a Lorentzian metric on $\mathcal{M}$. Let X be a Killing field on $(\mathcal{M}, g)$. Show also that, for any $V, W \in \Gamma(\mathcal{M})$:

$$g\big(\nabla_V X, W\big) + g\big(\nabla_W X, V\big) = 0,$$

where ∇ denotes the Levi-Civita connection associated to g (recall that ∇ is torsion-free and satisfies $\nabla_X g = 0$ for all $X \in \Gamma(\mathcal{M})$.) *(Hint: Apply the product rule on the expression $X(g(Y,Z)) = \mathcal{L}_X(g(Y,Z))$ for suitably chosen vector fields Y, Z.)*

Solution. (a) Using the formula $\mathcal{L}_X Y = [X,Y]$ holding for any $X, Y \in \Gamma(\mathcal{M})$, we can readily calculate that the relation

$$\mathcal{L}_{[X,Y]}Z = \mathcal{L}_X\mathcal{L}_Y Z - \mathcal{L}_Y\mathcal{L}_X Z$$

is equivalent to the statement that

$$\big[[X,Y],Z\big] = \big[X,[Y,Z]\big] - \big[Y,[X,Z]\big]$$

which, after rearranging the terms and using the anti-symmetry of $[\cdot,\cdot]$ in its arguments, is equivalent to

$$\big[[X,Y],Z\big] + \big[[X,Y],Z\big] + \big[[X,Y],Z\big] = 0.$$

The above is just Jacobi's identity.

Using the fact that, for any 1-form ω and any $X, Y \in \Gamma(\mathcal{M})$, the Lie derivative $\mathcal{L}_X\omega$ satisfies

$$X\big(\omega(Y)\big) = \mathcal{L}_X\omega(Y) + \omega\big(\mathcal{L}_X Y\big),$$

we can compute:

$$\mathcal{L}_X\omega(Y) = X\big(\omega(Y)\big) - \omega\big([X,Y]\big).$$

Therefore, we can readily calculate for any $X, Y, Z \in \Gamma(\mathcal{M})$

$$\begin{aligned}
\mathcal{L}_X(\mathcal{L}_Y\omega)(Z) &= X\big(\mathcal{L}_Y\omega(Z)\big) - \mathcal{L}_Y\omega([X,Z]) \\
&= X\big(Y(\omega(Z)) - \omega([Y,Z])\big) - Y\big(\omega([X,Z])\big) + \omega\big(\big[Y,[X,Z]\big]\big) \\
&= X\big(Y(\omega(Z))\big) - X\big(\omega([Y,Z])\big) - Y\big(\omega([X,Z])\big) + \omega\big(\big[Y,[X,Z]\big]\big)
\end{aligned}$$

and, after switching the roles of X, Y:

$$\mathcal{L}_Y(\mathcal{L}_X\omega)(Z) = Y\big(X(\omega(Z))\big) - Y\big(\omega([X,Z])\big) - X\big(\omega([Y,Z])\big) + \omega\big(\big[X,[Y,Z]\big]\big).$$

Subtracting the above relations (noting that the second and third term in each right hand side cancel out), we obtain

$$\begin{aligned}\mathcal{L}_X(\mathcal{L}_Y\omega)(Z)-\mathcal{L}_Y(\mathcal{L}_X\omega)(Z)&=[X,Y]\big(\omega(Z)\big)+\omega\big([Y,[X,Z]]-[X,[Y,Z]]\big)\\&=[X,Y]\big(\omega(Z)\big)-\omega\big([[X,Y],Z]\big)\\&=\big(\mathcal{L}_{[X,Y]}\omega\big)(Z)\end{aligned}$$

(where, in passing from the second to the third line above, we used Jacobi's identity). Since the above relation is true for any $Z\in\Gamma(\mathcal{M})$, we infer that

$$\mathcal{L}_{[X,Y]}\omega=\mathcal{L}_X\mathcal{L}_Y\omega-\mathcal{L}_Y\mathcal{L}_X\omega.$$

In order to prove that the same relation holds for any tensor field T, i.e.

$$\mathcal{L}_{[X,Y]}T=\mathcal{L}_X\mathcal{L}_YT-\mathcal{L}_Y\mathcal{L}_XT, \tag{1}$$

we can argue inductively on the type of T: If the formula is true for all tensor fields of type (k,l), then (due to linearity) (1) will also be true for all tensor fields of type $(k+1,l)$ if it's true for tensors of the form

$$T=\bar{T}\otimes V,$$

where $\bar{T}$ is of type (k,l) and $V\in\Gamma(\mathcal{M})$ (we get the same statement for tensor fields of type $(k,l+1)$ if we replace V with $\omega\in\Gamma^*(\mathcal{M})$). Using the formula

$$\mathcal{L}_XT=\mathcal{L}_X\bar{T}\otimes V+\bar{T}\otimes\mathcal{L}_XV,$$

verifying (1) using that it is true for $\bar{T}$ and V is a simple algebraic exercise. Similarly when V is replaced with $\omega\in\Gamma^*(\mathcal{M})$.

(b) If $X,Y\in\mathcal{K}$, then $\mathcal{L}_Xg=\mathcal{L}_Yg=0$. Using the commutator formula from part (a) of this exercise, we calculate

$$\mathcal{L}_{[X,Y]}g=\mathcal{L}_X\big(\mathcal{L}_Yg\big)-\mathcal{L}_Y\big(\mathcal{L}_Xg\big)=0-0=0.$$

(c) We will compute the expression $X\big(g(V,W)\big)$ in two different ways:

- Thinking in terms of the Lie derivative in the direction of X and using the fact that $\mathcal{L}_X(f\otimes h)=\mathcal{L}_Xf\otimes h+f\otimes\mathcal{L}_Xh$ and that $\mathcal{L}_X$ commutes with contractions, we obtain

  $$\begin{aligned}X\big(g(V,W)\big)=&\big(\mathcal{L}_Xg\big)(V,W)+g\big(\mathcal{L}_XV,W\big)+g\big(V,\mathcal{L}_XW\big)\\=&0+g\big([X,V],W\big)+g\big(V,[X,W]\big)\end{aligned}$$

  (note that we used our assumption that $\mathcal{L}_Xg=0$).

- Thinking in terms of the covariant derivative ∇_X, we have

  $$X\big(g(V,W)\big)=g\big(\nabla_XV,W\big)+g\big(V,\nabla_XW\big).$$

Combining the above relations, we obtain that the corresponding right hand sides must be equal to each other; using, in addition, the fact that the Levi-Civita connection ∇ is torsion-free, i.e.

$$\nabla_X Y - \nabla_Y X - [X,Y] = 0$$

we obtain:

$$\begin{aligned} g\big(\nabla_X V, W\big) + g\big(V, \nabla_X W\big) &= g\big([X,V],W\big) + g\big(V,[X,W]\big) \\ &= g\big(\nabla_X V, W\big) - g\big(\nabla_V X, W\big) + g\big(V, \nabla_X W\big) - g\big(V, \nabla_W X\big), \end{aligned}$$

i.e.:

$$g\big(\nabla_V X, W\big) + g\big(V, \nabla_W X\big) = 0. \tag{2}$$

4.2 (a) Let $F : (\mathbb{R}^{n+1}, \eta) \to (\mathbb{R}^{n+1}, \eta)$ be an isometry of Minkowski spacetime. Show that, with respect to the Cartesian coordinate system $(x^0, \dots, x^n)$ on $\mathbb{R}^{n+1}$, the differential dF satisfies at every $x \in \mathbb{R}^{n+1}$:

$$\eta_{\mu\nu}\partial_\alpha F^\mu(x)\partial_\beta F^\nu(x) = \eta_{\alpha\beta}.$$

Deduce that F is an *affine* isometry, i.e. takes the form $F(x) = Ax + b$. (*Hint: Differentiate the above relation.*)

(b) Let $t \to F_t : (\mathbb{R}^{n+1}, \eta) \to (\mathbb{R}^{n+1}, \eta)$ be a 1-parameter group of isometries. Prove that the generator of $\{F_t\}_{t\in\mathbb{R}}$, namely the vector field

$$x \to X|_x \doteq \frac{d}{dt}F_t(x)\Big|_{t=0},$$

is *affine*, i.e. is of the form

$$X^\mu(x) = A^\mu_\nu x^\nu + b^\mu$$

for some constants A^μ_ν, b^μ. Show also that the matrix A^μ_ν satisfies

$$\eta_{\alpha\mu}A^\mu_\beta + \eta_{\beta\mu}A^\mu_\alpha = 0. \tag{3}$$

What is the dimension of the Lie algebra of Killing vector fields on $(\mathbb{R}^{n+1}, \eta)$?

(c) Can you similarly classify all conformal Killing vector fields on $(\mathbb{R}^{n+1}, \eta)$ when $n \geqslant 2$?

(d) Show that the Lie group of isometries of the n-dimensional de Sitter space (see Exercise 2.1) has dimension at least $\frac{n(n+1)}{2}$ (we will later show that this is also an upper bound for the dimension of the group of isometries of any n-dimensional Lorentzian manifold).

Solution. (a) Since F is an isometry, the differential $dF : (T_p\mathbb{R}^{n+1}, \eta) \to (T_{F(p)}\mathbb{R}^{n+1}, \eta)$ is a linear isometry of vector spaces. Thus, we have:

$$\eta\big(dF(\frac{\partial}{\partial x^\alpha}), dF(\frac{\partial}{\partial x^\beta})\big) = \eta\big(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\big)$$

and, therefore (in view of the fact that $dF(\frac{\partial}{\partial x^\alpha}) = \partial_\alpha F^\mu \partial_\mu$), we obtain for any $\alpha, \beta = 0, \dots, n$:

$$\eta_{\mu\nu}\partial_\alpha F^\mu \partial_\beta F^\nu = \eta_{\alpha\beta}. \tag{4}$$

Differentiating the above relation with respect to ∂_γ (using the fact that the coefficients $\eta_{\alpha\beta}$ are constant in x and the symmetry of η), we get:

$$\begin{aligned}
0 &= \eta_{\mu\nu}\partial_\gamma\partial_\alpha F^\mu \partial_\beta F^\nu + \eta_{\mu\nu}\partial_\alpha F^\mu \partial_\gamma\partial_\beta F^\nu \\
&= \eta_{\mu\nu}\partial_\gamma\partial_\alpha F^\mu \partial_\beta F^\nu + \partial_\beta\big(\eta_{\mu\nu}\partial_\alpha F^\mu \partial_\gamma F^\nu\big) - \eta_{\mu\nu}\partial_\beta\partial_\alpha F^\mu \partial_\gamma F^\nu \\
&= \eta_{\mu\nu}\partial_\gamma\partial_\alpha F^\mu \partial_\beta F^\nu + \partial_\beta\big(\eta_{\mu\nu}\partial_\alpha F^\mu \partial_\gamma F^\nu\big) - \partial_\alpha\big(\eta_{\mu\nu}\partial_\beta F^\mu \partial_\gamma F^\nu\big) + \eta_{\mu\nu}\partial_\beta F^\mu \partial_\alpha\partial_\gamma F^\nu \\
&= \eta_{\mu\nu}\partial_\gamma\partial_\alpha F^\mu \partial_\beta F^\nu + \partial_\beta\big(\eta_{\alpha\gamma}\big) - \partial_\alpha\big(\eta_{\beta\gamma}\big) + \eta_{\mu\nu}\partial_\beta F^\mu \partial_\alpha\partial_\gamma F^\nu \\
&= 2\eta_{\mu\nu}\partial_\gamma\partial_\alpha F^\mu \partial_\beta F^\nu.
\end{aligned}$$

Because η is non-degenerate and dF is invertible (as a linear isometry), we infer that the matrix $M_{\nu\alpha} = \eta_\mu\nu\partial_\alpha F^\mu$ (which is simply the matrix-product of the matrices $[\eta_{\mu\nu}]$ and $[\partial_\alpha F^\mu]$) is invertible; hence, by multiplying the above relation with $N^{\lambda\nu}$ (where $N = M^{-1}$), we infer that

$$\partial_\gamma\partial_\beta F^\lambda = 0 \quad \text{for all } \beta, \gamma, \lambda = 0, \dots n.$$

Therefore, F^λ has to be an affine function, i.e. $F^\lambda(x) = A^\lambda_\mu x^\mu + b^\lambda$ for some constants A^λ_μ and b^λ, $\mu, \lambda = 0, \dots, n$. Note that, in this case, $(dF)^\beta_\alpha = A^\beta_\alpha$; hence (in view of the fact that dF is a linear isometry), the matrix A is orthonormal with respect to η (i.e. belongs to the group $SO(1,n)$).

(b) If $t \to F_t : (\mathbb{R}^{n+1}, \eta) \to (\mathbb{R}^{n+1}, \eta)$ is an 1-parameter group of isometries, then, in view of part (a), it takes the form (in the $(x^0, \dots, x^n)$ Cartesian coordinate system):

$$(F_t)^\mu(x) = (C_t)^\mu_\nu x^\nu + (d_t)^\mu. \tag{5}$$

Differentiating[1] (5) with respect to t and setting $A^\nu_\mu \doteq \frac{d}{dt}(C_t)^\mu_\nu\Big|_{t=0}$ and $b^\mu \doteq \frac{d}{dt}(d_t)^\mu\Big|_{t=0}$, we obtain that the vector field $X \in \Gamma(\mathbb{R}^{n+1})$ which is the generator of the family F_t, i.e. defined at every point $x \in \mathbb{R}^n$ as $X|_x = \frac{d}{dt}F_t(x)\big|_{t=0}$, satisfies

$$X^\mu(x) = A^\mu_\nu x^\nu + b^\mu. \tag{6}$$

The relation (4) for F_t reads

$$\eta_{\mu\nu}(C_t)^\mu_\alpha (C_t)^\nu_\beta = \eta_{\alpha\beta}.$$

Differentiating the above relation at $t = 0$ and using the fact that $A^\nu_\mu \doteq \frac{d}{dt}(C_t)^\mu_\nu\Big|_{t=0}$ and $(C_0)^\mu_\nu = \delta^\mu_\nu$ (since $F_0(x) = x$, as a consequence of the fact that F_t is a 1-parameter semigroup), we obtain:

$$\eta_{\mu\beta}A^\mu_\alpha + \eta_{\alpha\nu}A^\nu_\beta = 0,$$

which can be reexpressed (after relabelling the indices) as (3).

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[1] The fact that $F_t(x)$ (and, as a consequence, $(C_t)^\mu_\nu$ and $(d_t)^\mu$) is differentiable in t follows from our assumption that F_t is an 1-parameter group of diffeomorphisms.

Remark. Note that (3) is nothing else than the usual equation $\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0$ satisfied by a Killing vector field on a general Lorentzian manifold $(\mathcal{M}, g)$ (see Ex. 4.1.c).

If we define

$$\tilde{A}_{\alpha\beta} \doteq \eta_{\mu\alpha} A^\mu_\beta,$$

then (3) is equivalent to the statement that $\tilde{A}_{\alpha\beta}$ is antisymmetric. As a result, $\tilde{A}$ has at most $\frac{n(n+1)}{2}$ degrees of freedom (as many as an antisymmetric $(n+1)\times(n+1)$ matrix); since A (as a matrix) is obtained from $\tilde{A}$ by multiplication with an invertible matrix, we infer that A has at most $\frac{n(n+1)}{2}$ degrees of freedom too. As a result, the right hand side of (6) has at most $\frac{n(n+1)}{2} + n + 1 = \frac{(n+1)(n+2)}{2}$ linearly independent free parameters (corresponding to the free parameters in A and b) and, hence, the space of Killing vector fields in Minkowski spacetime is at most $\frac{(n+1)(n+2)}{2}$-dimensional. One can verify that its dimension is exactly $\frac{(n+1)(n+2)}{2}$ by noting that the generators translations, spatial rotations and hyperbolic rotations (boosts) form a set of $\frac{(n+1)(n+2)}{2}$ linearly independent Killing vector fields, namely

$$\begin{aligned}
T_\alpha &= \frac{\partial}{\partial x^\alpha}, \quad \alpha \in \{0, \dots, n\},\\
\Omega_{ij} &= x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}, \quad i < j \in \{1, \dots, n\},\\
\Omega_{0i} &= x^0 \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial x^0}, \quad i \in \{1, \dots, n\}.
\end{aligned}$$

(c) Recall that a conformally Killing vector field X on a Lorentzian manifold $(\mathcal{M}, g)$ satisfies

$$\mathcal{L}_X g = f \cdot g$$

for some $f \in C^\infty(\mathcal{M})$. The analogue of the Killing equation (2) then becomes (with essentially the same derivation)

$$g(\nabla_V X, W) + g(V, \nabla_X W) = f \cdot g(V, W) \quad \text{for all } V, W \in C^\infty(\mathcal{M}).$$

If V, W are coordinate vector fields in a given coordinate chart $(x^0, \dots, x^n)$, the above relation yields (recalling our convention that

$$X_\alpha \doteq g_{\alpha\beta} X^\beta$$

and the fact that the musical isomorphism commutes with covariant differentiation)

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = f g_{\mu\nu}.$$

In the case that we are interested in, namely when $(\mathcal{M}, g) = (\mathbb{R}^{n+1}, \eta)$ and $(x^0, \dots, x^n)$ are the standard Cartesian coordinates on $\mathbb{R}^{n+1}$, the above relation becomes

$$\partial_\mu X_\nu + \partial_\nu X_\mu = f \eta_{\mu\nu}. \tag{7}$$

Contracting the above equation with $\eta^{\mu\nu}$, we infer that

$$2\partial^\lambda X_\lambda \doteq 2\eta^{\mu\nu}\partial_\mu X_\nu = (n+1) f \quad \Rightarrow \quad f = \frac{2}{n+1}\partial^\lambda X_\lambda.$$

Setting $divX \doteq \partial^\lambda X_\lambda$, we can substitute for f as above in (7) to obtain:

$$\partial_\mu X_\nu + \partial_\nu X_\mu = \frac{2}{n+1}(divX)\eta_{\mu\nu}. \tag{8}$$

We will use (8) to exchange the indices in expressions of the form $\partial_\mu X_\nu$. In particular, we calculate:

$$\begin{aligned}\partial_\alpha\partial_\beta\partial_\mu X_\nu &\overset{(8)}{=} -\partial_\alpha\partial_\beta\partial_\nu X_\mu + \frac{2}{n+1}\eta_{\mu\nu}\partial_\alpha\partial_\beta divX\\ &= -\partial_\alpha\partial_\nu\partial_\beta X_\mu + \frac{2}{n+1}\eta_{\mu\nu}\partial_\alpha\partial_\beta divX\\ &\overset{(8)}{=} \partial_\alpha\partial_\nu\partial_\mu X_\beta - \frac{2}{n+1}\eta_{\mu\beta}\partial_\alpha\partial_\nu divX + \frac{2}{n+1}\eta_{\mu\nu}\partial_\alpha\partial_\beta divX\\ &\overset{(8)}{=} -\partial_\alpha\partial_\beta\partial_\mu X_\nu + \frac{2}{n+1}\eta_{\beta\nu}\partial_\alpha\partial_\mu divX - \frac{2}{n+1}\eta_{\mu\beta}\partial_\alpha\partial_\nu divX + \frac{2}{n+1}\eta_{\mu\nu}\partial_\alpha\partial_\beta divX\end{aligned} \tag{9}$$

and, after moving the first term on the right hand side to the left hand side (noting that they are the same term), we infer:

$$\partial_\alpha\partial_\beta\partial_\mu X_\nu = \frac{1}{n+1}\eta_{\beta\nu}\partial_\alpha\partial_\mu divX - \frac{1}{n+1}\eta_{\mu\beta}\partial_\alpha\partial_\nu divX + \frac{1}{n+1}\eta_{\mu\nu}\partial_\alpha\partial_\beta divX. \tag{10}$$

From the above equation, we can obtain a relation for $divX = \eta^{\alpha\nu}\partial_\alpha X_\nu$ by taking the contraction with respect to $\eta^{\alpha\nu}$, which yields (recall that $\eta^{\kappa\lambda}\eta_{\lambda\rho} = \delta^\kappa_\rho$):

$$\partial_\beta\partial_\mu divX = \frac{2}{n+1}\partial_\beta\partial_\mu divX - \eta_{\mu\beta}\Box_\eta(divX) \quad \Rightarrow \quad \partial_\beta\partial_\mu divX = \frac{n+1}{n-1}\eta_{\mu\beta}\Box_\eta(divX), \tag{11}$$

where we denote $\Box_\eta h \doteq \eta^{\gamma\delta}\partial_\gamma\partial_\delta h$ (this is the usual wave operator).

Remark. Note that, in the above, it is crucial that $n > 1$; in the $1+1$ dimensional case, the above identity becomes trivial.

Taking one more contraction of (11) with $\eta^{\beta\mu}$, we finally obtain the following relation for $\Box_\eta divX$:

$$\Box_\eta divX = \frac{(n+1)^2}{n-1}\Box_\eta(divX) \quad \Rightarrow \quad \Box_\eta(divX) = 0.$$

Substituting the above in (11), we obtain:

$$\partial_\beta\partial_\mu divX = 0.$$

Substituting this in (10), we finally obtain that a conformally Killing vector field on $(\mathbb{R}^{n+1}, \eta)$ for $n \geqslant 2$ satisfies

$$\partial_\alpha\partial_\beta\partial_\mu X_\nu = 0,$$

i.e. the components of X are at most quadratic in the cartesian coordinates:

$$X_\nu = M_{\nu\alpha\beta}x^\alpha x^\beta + B_{\nu\alpha}x^\alpha + b_\nu \tag{12}$$

(where the $M_{\nu\alpha\beta}$, $B_{\nu\alpha}$, b_ν are constants and $M_{\nu\alpha\beta}$ is symmetric in α, β).

In order to determine the relations satisfied by the coefficients in (12), we can substitute in the first order relation (8), from which we obtain

$$B_{\mu\nu} + B_{\nu\mu} = \frac{2}{n+1}\eta^{\alpha\beta}B_{\alpha\beta}\eta_{\mu\nu} \tag{13}$$

(note that the above relation is satisfied for any $B_{\alpha\beta}$ which is of the form "antisymmetric"$+\lambda\eta_{\mu\nu}$) and

$$M_{\nu\mu\alpha} + M_{\mu\nu\alpha} = \frac{2}{n+1}\eta^{\gamma\delta}M_{\gamma\delta\alpha}\eta_{\mu\nu} \quad \text{for all } \ \alpha = 0, \dots, n.$$

Note that, similarly to (13), the above relation only admits solutions of the form $M^{(ant)}_{\mu\nu\alpha} + \lambda_\alpha \eta_{\mu\nu}$, where $M^{(ant)}_{\mu\nu\alpha}$ is antisymmetric in μ, ν. The solutions of this form that are also symmetric in ν, α are then completely specified by λ_α: We have

$$\begin{aligned}
M^{(ant)}_{\mu\nu\alpha} + \lambda_\alpha \eta_{\mu\nu} &= M^{(ant)}_{\mu\alpha\nu} + \lambda_\nu \eta_{\mu\alpha} \\
&= -M^{(ant)}_{\alpha\mu\nu} + \lambda_\nu \eta_{\mu\alpha} \\
&= -\big(M^{(ant)}_{\alpha\mu\nu} + \lambda_\nu \eta_{\alpha\mu}\big) + 2\lambda_\nu \eta_{\mu\alpha} \\
&= -\big(M^{(ant)}_{\alpha\nu\mu} + \lambda_\mu \eta_{\alpha\nu}\big) + 2\lambda_\nu \eta_{\mu\alpha} \\
&= M^{(ant)}_{\nu\alpha\mu} - \lambda_\mu \eta_{\alpha\nu} + 2\lambda_\nu \eta_{\mu\alpha} \\
&= \big(M^{(ant)}_{\nu\alpha\mu} + \lambda_\mu \eta_{\nu\alpha}\big) - 2 - \lambda_\mu \eta_{\alpha\nu} + 2\lambda_\nu \eta_{\mu\alpha} \\
&= \big(M^{(ant)}_{\nu\mu\alpha} + \lambda_\alpha \eta_{\nu\mu}\big) - 2\lambda_\mu \eta_{\alpha\nu} + 2\lambda_\nu \eta_{\mu\alpha} \\
&= -M^{(ant)}_{\mu\nu\alpha} + \lambda_\alpha \eta_{\nu\mu} - 2\lambda_\mu \eta_{\alpha\nu} + 2\lambda_\nu \eta_{\mu\alpha}
\end{aligned}$$

from which we obtain (after moving the first term in the right hand side to the left hand side):

$$M^{(ant)}_{\mu\nu\alpha} = \lambda_\nu \eta_{\mu\alpha} - \lambda_\mu \eta_{\alpha\nu}.$$

From all the above, we infer that the dimension of the Lie algebra of conformally Killing vector fields on $(\mathbb{R}^{n+1}, \eta)$ has dimension $(n+1) + \frac{(n+1)(n+2)}{2} + (n+1) = \frac{(n+1)(n+6)}{2}$. A basis of conformally Killing vector fields consists of the basis of Killing vector fields presented in part b, plus the dilation vector field $S = x^\alpha \partial_\alpha$ and the generators of the *special conformal transformations*, a basis of which are the vector fields $K_{(\lambda)} = 2\eta_{\lambda\alpha}x^\alpha x^\nu \partial_\nu - \eta_{\alpha\beta}x^\alpha x^\beta \partial_\lambda$, $\lambda = 0, \dots, n$.

Remark. In $1+1$ dimensions, the Lie algebra of conformally Killing vector fields is *infinite* dimensional. Recall that this case is special in other ways as well with respect to the conformal structure: Any $1+1$ dimensional Lorentzian manifold is locally conformally equivalent to Minkowski spacetime.

(d) Recall that n-dimensional de-Sitter space is isometrically embedded in $(\mathbb{R}^{n+1}, \eta)$ as the submanifold $S = \{x \in \mathbb{R}^{n+1} : \eta_{\alpha\beta}x^\alpha \beta = +1\}$ equipped with the induced (Lorentzian) metric. As a result, any isometry of $(\mathbb{R}^{n+1}, \eta)$ that maps S to itself should give rise to an isometry of S. It can be readily seen that every linear isometry (i.e. fixing the origin) of $(\mathbb{R}^{n+1}, \eta)$ maps S to itself, since, due to linearity, for any such isometry F we must have $\eta(F(x), F(x)) = \eta(x, x)$ (treating $x \in \mathbb{R}^{n+1}$

as a vector). In part (b), we saw that the group of such isometries of $(\mathbb{R}^{n+1},\eta)$ has dimension $\frac{n(n+1)}{2}$ (generated by the Killing vector fields with vanishing constant term b). Thus, the group of isometries of de Sitter space has dimension at least equal to this.

4.3 In this exercise, we will study the question of whether a given Lorentzian manifold can be extended, i.e. isometrically embedded into a larger Lorentzian manifold in a non trivial way. This question arises often in general relativity in the study of singularities emerging from smooth "initial data" (such as the singularities in the interior of black holes, or "big bang"-type singularities).

(a) Consider the spacetime $(\mathcal{M},g)$ with $\mathcal{M}=\mathbb{R}_t\times(x_0,+\infty)_x$, $x_0>0$, and

$$g=-\big(1-\frac{x_0}{x}\big)dt^2+\big(1-\frac{x_0}{x}\big)^{-1}dx^2.$$

Show that $(\mathcal{M},g)$ can be extended smoothly "beyond $x=x_0$" by finding a coordinate transformation $(t,x)\to(\tilde{t}(t,x),x)=(t+f(x),x)$ such that, with respect to the new coordinates, the components of g can be smoothly extended as functions of $(\tilde{t},x)$ beyond $x=x_0$. Can you express this process as an embedding of $(\mathcal{M},g)$ into a larger Lorentzian manifold?

*(b) Consider the spacetime $(\mathcal{N},h)$ with $\mathcal{N}=(0,+\infty)_t\times\mathbb{S}^1_\theta$ and

$$h=-tdt^2+\frac{1}{t}d\theta^2.$$

We will show that $(\mathcal{N},h)$ is inextendible as a C^0 Lorentzian manifold beyond $t=0$. We will achieve this in a number of steps; we will essentially follow the method introduced by Sbierski to prove a similar statement for the interior of the Schwarzschild black hole.

1. For any Lorentzian manifold $(\mathcal{M}',g')$ and any domain $\mathcal{U}$, we will define the *spacelike diameter* of $\mathcal{U}$ by

$$\text{spdiam}\mathcal{U}\doteq\sup\Big\{\ell(\gamma):\gamma\subset\mathcal{U}\text{ is a spacelike curve }\Big\}.$$

Show that if $p,q\in(\mathbb{R}^{1+1},\eta)$ with $q\in I^-(p)$, then

$$\text{spdiam}I^+(q)\cap I^-(p)\leqslant\sqrt{(p^0-q^0)^2-(p^1-q^1)^2}.$$

2. If g is a C^0 oriented Lorentzian metric on $\mathbb{R}^{1+1}$, show that, for every pairs of sequences of points $p_n,q_n\in\mathbb{R}^n$ with $q_n\in I^-(p_n)$ (the past cone defined with respect to g) and $\lim_{n\to+\infty}q_n=\lim_{n\to+\infty}p_n=p$, we have

$$\text{spdiam}I^+(q_n)\cap I^-(p_n)\xrightarrow{n\to+\infty}0.$$

3. We will assume as given the following statement:
If $(\mathcal{M}', g')$ is a $1+1$ dimensional and oriented C^0 Lorentzian manifold and $\Omega \subset \mathcal{M}'$ is a domain in $\mathcal{M}'$ with the property that Ω covers the past of $\partial\Omega$, i.e. $I^-(p) \subset \Omega$ for any $p \in \partial\Omega$, then the following holds: For any $p \in \partial\Omega$, there exists a sequence $q_n \in I^-(p)$ and $p_n \in I^+(p)$ such that $q_n, p_n \xrightarrow{n\to+\infty} p$ and $I^+(q_n) \cap \Omega \subset I^-(p_n)$.
Using that, show that $(\mathcal{N}, h)$ above is not extendible as a C^0 spacetime beyond $t = 0$.

Solution. (a) Setting $\tilde{t}(t,x) = t + f(x)$ (for a function $f : (x_0, +\infty) \to \mathbb{R}$ to be determined), we compute that

$$d\tilde{t} = dt + f' dx$$

and, therefore

$$\begin{aligned}
g &= -\big(1 - \frac{x_0}{x}\big)dt^2 + \big(1 - \frac{x_0}{x}\big)^{-1}dx^2 \\
&= -\big(1 - \frac{x_0}{x}\big)(d\tilde{t} - f'dx)^2 + \big(1 - \frac{x_0}{x}\big)^{-1}dx^2 \\
&= -\big(1 - \frac{x_0}{x}\big)d\tilde{t}^2 + 2\big(1 - \frac{x_0}{x}\big)f'(x)d\tilde{t}dx + \Big(\big(1 - \frac{x_0}{x}\big)^{-1} - \big(1 - \frac{x_0}{x}\big)(f'(x))^2\Big)dx^2.
\end{aligned}$$

Therefore, defining $f(x)$ so that $f'(x) = \big(1 - \frac{x_0}{x}\big)^{-1}$ near $x = x_0$ (i.e. $f(x) = \log(x - x_0) + h(x)$ for some $h(x)$ which is smooth and bounded at $x = x_0$), we infer that the components of g in the $(\tilde{t}, x)$ coordinate chart can be smoothly extended (as functions) across x_0. Let us denote with $\tilde{g}$ such an extension on $(x_0 - \epsilon, +\infty)$ for some $\epsilon > 0$. Identifying in the $(\tilde{t}, x)$ coordinates $\mathcal{M}$ with the domain $\mathbb{R} \times (x_0, +\infty)$ and setting $\tilde{\mathcal{M}} = \mathbb{R} \times (x_0 - \epsilon, +\infty)$, we infer that the inclusion $\mathbb{R} \times (x_0, +\infty) \hookrightarrow \mathbb{R} \times (x_0 - \epsilon, +\infty)$ defines an isometric embedding $(\mathcal{M}, g) \to (\tilde{\mathcal{M}}, \tilde{g})$.

(b) 1. The notion of spacelike diameter is purely geometric, i.e. it is preserved under isometric transformations. Moreover, the bound that we need to show, namely

$$\mathrm{spdiam}\Big(I^+(q) \cap I^-(p)\Big) \leqslant \sqrt{(p^0 - q^0)^2 - (p^1 - q^1)^2}$$

is also invariant under isometric transformations, since the right hand side is simply the timelike distance between p and q. Therefore, without loss of generality (by possibly applying a translation and a boost on $(\mathbb{R}^{1+1}, \eta)$, if necessary), we can assume that the points p and q have coordinates (in the (t,x) Cartesian coordinate system)

$$p = (T, 0), \quad q = (-T, 0)$$

for some $T > 0$, so that $I^+(q) \cap I^-(p)$ becomes the square

$$I^+(q) \cap I^-(p) = \big\{(t,x) : \, -T \leqslant t - x \leqslant T, \, -T \leqslant t + x \leqslant T\big\}.$$

We will show that any spacelike curve in the above set has length at most equal to $2T$, thus proving that the spacelike diameter of $I^+(q) \cap I^-(p)$ is at most $2T$ (in fact, it is equal to this value, since the maximal length is achieved by the straight line segment connecting $(0, -T)$ to $(0, T)$). Let

$\gamma : [a,b] \to I^+(q) \cap I^-(p)$ be a spacelike curve. Since $\eta = -dt^2 + dx^2$, the condition that γ is spacelike translates to $|\dot{\gamma}^x| > |\dot{\gamma}^t| \geqslant 0$. We can therefore reparametrize γ so that $\dot{\gamma}^x = 1$ (i.e. parametrize it by the x coordinate); recall that the length of the curve is invariant under reparametrizations. In this case, $\gamma(x)$ takes the form

$$\gamma(x) = (t(x), x), \quad x_0 \leqslant x \leqslant x_1$$

for some $x_0 < x_1$ in $[-T, T]$, with the condition that γ is spacelike translating to $\left|\frac{dt}{dx}\right| < 1$. Thus, we compute:

$$\ell(\gamma) = \int_{x_0}^{x_1} \sqrt{\eta(\dot{\gamma}, \dot{\gamma})}\, dx = \int_{x_0}^{x_1} \sqrt{1 - \big(\frac{dt}{dx}\big)^2}\, dx \leqslant \int_{x_0}^{x_1} dx = x_1 - x_0 \leqslant 2T.$$

Thus, we have shown that

$$\text{spdiam}\Big(I^+(q) \cap I^-(p)\Big) \leqslant 2T = \text{timelike distance of } p, q.$$

2. Without loss of generality, we will assume that we have fixed a global coordinate system (t,x) on $\mathbb{R}^{1+1}$ such that $p - (0,0)$ and ∂_t, ∂_x are orthonormal at the point p, i.e.

$$g|_p = -dt^2 + dx^2$$

(note that this can always be achieved by an affine change of coordinates). Since g is a C^0 Lorentzian metric, we infer that, near $p = (0,0)$, we the components of g behave as follows:

$$g = (-1 + o(1))dt^2 + o(1)dtdx + (1 + o(1))dx^2. \tag{14}$$

Moreover, since $(\mathbb{R}^{1+1}, g)$ is time oriented, we can fix a globally defined future directed timelike vector field V and we will assuming (by applying the transformation $t \to -t$, if necessary) that, at the point p, $\partial_t|_p$ is future directed. For any $\rho > 0$, we will also denote with B_ρ the coordinate (open) ball of radius ρ around p, i.e.

$$B_\rho \doteq \big\{(t,x) : \sqrt{t^2 + x^2} < \rho\big\}.$$

Let $\delta_0 > 0$ be sufficiently small (depending only on the geometry of $(\mathbb{R}^{1+1}, g)$ near p). For any $\delta \in (0, \delta_0)$, if $y, z \in B_\delta$ with y lying in the past of z, then

$$I^+(y) \cap I^-(z) \subset B_{\sqrt{\delta}}. \tag{15}$$

Informally, the above inclusion says that if y, z are both sufficiently close to p, then any future directed timelike curve from y to z also has to stay close to p. We will show (15) below; for now, let us assume that (15) holds, so that

$$\text{spdiam}\Big(I^+(y) \cap I^-(z)\Big) \leqslant \text{spdiam}\big(B_{\sqrt{\delta}}\big).$$

We will show that, if δ_0 is small enough so that the $o(1)$ terms in (14) are of size $\leqslant \frac{1}{10}$ on $B_{2\sqrt{\delta_0}}$, we have

$$\text{spdiam}\big(B_{\sqrt{\delta}}\big) \leqslant 4\sqrt{\delta}, \tag{16}$$

from which it follows that $\lim_{n\to+\infty} q_n = \lim_{n\to+\infty} p_n = p$ with $q_n \in I^-(p_n)$,then

$$\text{spdiam} I^+(q_n) \cap I^-(p_n) \xrightarrow{n\to+\infty} 0.$$

In order to prove (16), we will argue similarly as for part 2: If γ is a spacelike curve $B_{\sqrt{\delta}}$, then, in view of (14), we have $|\dot\gamma^x| > \frac{1}{2}|\dot\gamma^t|$ and hence γ can be parametrized by x, so that $\gamma(x) = (t(x), x)$. Then, we have (recall that we assumed that the $o(1)$ terms in (14) are of size $\leqslant \frac{1}{10}$ on $B_{2\sqrt{\delta_0}}$):

$$\ell(\gamma) = \int_{x_0}^{x_1} \sqrt{g(\dot\gamma,\dot\gamma)}\,dx = \int_{x_0}^{x_1} \sqrt{-(1+o(1))|t'|^2 + o(1)t' + (1+o(1))}\,dx \leqslant \int_{x_0}^{x_1} 2\,dx \leqslant 2 \sup_{(x_0,t_0),(x_1,t_1)\in B_{\sqrt{\delta}}} \big|x_1 - x$$

Thus, (16) holds.

Proof of (15)**.** We will establish (15) by contradiction: Assume that there exists a *future directed timelike* curve $\gamma : [0,A] \to \mathbb{R}^{1+1}$ such that $\gamma(0) = y$, $\gamma(A) = z$ and $\gamma(s_0) \notin B_{\sqrt{\delta}}$ for some $s_0 \in (0,A)$.

Provided $\delta_0 > 0$ was chosen sufficiently small in terms of the geometry of $(\mathbb{R}^{1+1}, g)$, the coordinate ball $B_{2\sqrt{\delta}}$ will lie inside the coordinate system (t,x) fixed above around p and that the $o(1)$ terms in (14) are of size $\leqslant \frac{1}{10}$. By reparametrizing the domain of γ if necessary, we can assume without loss of generality that $\gamma \cap B_{\sqrt{\delta}}$ is parametrized by $t + \text{const}$, i.e. that in $\gamma \cap B_{\sqrt{\delta}}$, we have $\dot\gamma^t = 1$ and, thus, since γ is timelike (and $|g_{\alpha\beta} - \eta_{\alpha\beta}| \leqslant \frac{1}{10}$), we must have

$$|\dot\gamma^x| < \frac{3}{2} \quad \text{on} \quad \gamma \cap B_{\sqrt{\delta}}. \tag{17}$$

Let $s_* \in (0, s_0]$ be the largest number such that $\gamma(s) \in B_{\sqrt{\delta}}$ for all $s \in [0, s_*)$ (note that $s_* \leqslant s_0$ since $\gamma(s_0) \notin B_{\sqrt{\delta}}$). Note that, by continuity, we should have $\gamma(s_*) \in \text{clos}B_{\sqrt{\delta}} \setminus B_{\sqrt{\delta}}$, so that

$$t^2|_{\gamma(s_*)} + x^2|_{\gamma(s_*)} = \delta. \tag{18}$$

In view of the fact that $\dot\gamma^t = 1$ for $s \in [0, s_*)$, we have

$$\big|t|_{\gamma(s_*)} - t|_{\gamma(0)}\big| = |s_*|.$$

In view of the fact that (17) holds for $s \in [0, s_*)$, we have

$$\big|x|_{\gamma(s_*)} - x|_{\gamma(0)}\big| \leqslant \frac{3}{2}|s_*|.$$

Those estimates, combined with the fact that $\gamma(0) \in B_\delta$ (and hence $\big|t|_{\gamma(0)}\big|, \big|x|_{\gamma(0)}\big| \leqslant \delta$) and $\delta \ll \frac{1}{100}\sqrt{\delta}$ (which holds if δ_0 has fixed sufficiently small), imply, in view of (18), that

$$|s_*| \geqslant \frac{1}{4}\sqrt{\delta}.$$

Let us now consider the curve $\tilde\gamma : [0,A] \to \mathbb{R}^{1+1}$ such that

$$\tilde\gamma(s) = \gamma(s) \quad \text{for} \quad s \in [s_*, A]$$

while $\gamma|_{[0,s_c]}$ is contained in $\mathrm{clos}(B_{\sqrt{\delta}})$ and is defined (in the (t,x) coordinates) as the straight line segment connecting $\gamma(A)=z$ to $\gamma(s_*)$, i.e.

$$\tilde{\gamma}(s)=(\frac{s}{s_*}t|_{\gamma(s_*)}+\frac{s_*-s}{s_*}t|_{\gamma(A)},\ \frac{s}{s_*}x|_{\gamma(s_*)}+\frac{s_*-s}{s_*}x|_{\gamma(A)}).$$

Note that $\tilde{\gamma}$ is *closed*, piecewise C^1 and satisfies

$$\dot{\tilde{\gamma}}(s)=\dot{\gamma}(s)\quad \text{for} s\in(s_*,A)$$

and (since $|t(\gamma(A))|,|x(\gamma(A))|\leqslant\delta\ll\sqrt{\delta}$)

$$\dot{\tilde{\gamma}}^t(s)>\frac{1}{2}|\dot{\tilde{\gamma}}^x|(s)\quad\text{for}\quad s\in(0,s_*).$$

Let $\sigma:[0,A]\to\mathbb{R}^{1+1}$ be a smoothing out of $\tilde{\gamma}$ so that σ is still a closed curve, is pointwise δ-close to $\tilde{\gamma}$ and satisfies

$$\dot{\sigma}(s)=\dot{\tilde{\gamma}}(s)\quad\text{for}\quad s\in[s_*+\delta,A-\delta]$$

(hence $\dot{\sigma}(s)$ is timelike and future directed in the above interval) and

$$\dot{\sigma}(s)=\dot{\gamma}(s))>\frac{1}{3}|\dot{\tilde{\gamma}}^x|(s)\geqslant 0\quad\text{for}\quad s\in[0,s_*+\delta]\cup[A-\delta,A]. \tag{19}$$

Note that σ might still have self-intersections. We, can however, remove those self-intersections: If $s^{(0)}<s^{(1)}\in(0,A)$ are two points such that $\sigma(s^{(0)})=\sigma(s^{(1)})$, then we can remove the interval $(s^{(0)},s^{(1)})$ from the domain of σ and the remaining curve will still be closed, continuous and piecewise C^1; we can then smooth-out again this curve, and the tangent to the new curve will still satisfy the above estimates (if $\dot{\sigma}(s)$ was originally timelike and future directed near $s=s_0$, the smoothing will retain this property, since the set of future directed timelike vectors at a point forms an open *convex* cone; similarly for the condition (19)). Thus, after repeating the above "surgery" procedure a finite number of times, we are left with a *simple closed curve* σ.

We will now show that the existence of such a simple closed curve leads to a contradiction in view of the fact that g admits a global timelike vector V, based on a topological argument (which uses the fact that the background manifold is $\mathbb{R}^2$). Note that the above conditions on $\dot{\sigma}$ and the fact that V is timelike and future directed (and hence, $V^t>\frac{9}{10}|V^x|\geqslant 0$ on $B_{2\sqrt{\delta}}$) imply that

$$\lambda V|_{\sigma(s)}+\lambda'\dot{\sigma}(s)\neq 0\quad\text{for all }\ s\in[0,A]\ \text{ and }\ \lambda,\lambda'\geqslant 0\ \text{ with }(\lambda,\lambda')\neq(0,0)$$

(for the region $s\in[s_*+\delta,A-\delta]$, the above follows from the fact that both vectors are future directed and timelike; for $s\in[0,s_*+\delta]\cup[A-\delta,A]$, just check the t-components of V and $\dot{\sigma}$ are positive). Thus, if X is an arbitrary extension of the vector field $\dot{\sigma}$ to an open neighborhood $\mathcal{U}_\sigma$ of σ (such an extension exists because σ was assumed to not have self-intersections) then, if $\mathcal{V}_\sigma$ is an even smaller open neighborhood of σ (sufficiently small in terms of the precise form of σ, the extension X and the vector field V) and $f:R^{1+1}\to[0,1]$ is a smooth function such that

$$f|_\sigma=1\quad\text{and}\quad f=0\ \text{ on }\ \mathbb{R}^{1+1}\setminus\mathcal{V}_\sigma,$$

the vector field

$$W = f \cdot X + (1-f) \cdot V$$

is nowhere 0 on $\mathbb{R}^{1+1}$ and is tangent to the simple closed curve σ. However, such a curve σ bounds a closed topological disc D in $\mathbb{R}^2$ (by the Jordan curve theorem); this is a contradiction, since any smooth vector field on D which is tangent to ∂D must vanish at least once in D (otherwise, if we glued two such copies of D along ∂D we would get a non-vanishing continuous vector field on $\mathbb{S}^2$).

3. Let us fix a time orientation on $(\mathcal{N}, h)$ so that $-\partial_t$ is future directed (so that $t=0$ lies "to the future" of $\mathcal{N} = (0,+\infty)_t \times \mathbb{S}^1_\theta$). Assume, for the sake of contradiction, that $(\mathcal{N}, h)$ admits an extension as a C^0 Lorentzian manifold beyond $t=0$, i.e. that there exists a C^0 Lorentzian manifold $(\mathcal{N}', h')$ and an open domain $\Omega \subset \mathcal{N}'$ with $\Omega \neq \mathcal{N}'$ such that $(\mathcal{N}, h)$ can be isometrically identified with (Ω, h') and, under this identification, the function t on $\mathcal{N}$ (i.e. on Ω) can be continuously extended up to $\partial\Omega$ with $t|_{\partial\Omega} = 0$. Note that, for any $c > 0$, we have that the level set $\{t = c\}$ is entirely contained in Ω (i.e. $\mathcal{N}$), it is a spacelike hypersurface and, in view of the explicit form of h, we have that, for any $p \in \{t = c\}$, the past of p satisfies $I^-(p) \subset \{t > c\}$. By continuity of h' across $\partial\Omega$, we infer that the tangent space of $\partial\Omega = \{t = 0\}$ at every point is either spacelike or null (as the limit of spacelike hypersurfaces) and, for any $p \in \partial\Omega$, we have $I^-(p) \subseteq \{t > 0\} = \Omega$ (note that the limitting process at first appears to only guarantee that $I^-(p) \subset \mathrm{clos}\Omega$; however, $I^-(p) \cap \partial\Omega = \emptyset$, since $I^-(p)$ is an open set of $\mathcal{N}'$ and hence, if it contains a point on $\partial\Omega$, it should also intersect $\mathcal{N}' \setminus \Omega$). Thus, Ω covers the past of $\partial\Omega$.

Let $p \in \partial\Omega$. There exists a sequence $q_n \in I^-(p)$ and $p_n \in I^+(p)$ such that $q_n, p_n \xrightarrow{n\to+\infty} p$ and $I^+(q_n) \cap \Omega \subset I^-(p_n)$ (see the statement of the exercise). Since $(\mathcal{N}, h')$ is a C^0 Lorentzian manifold and p has a small neighborhood diffeomorphic to $\mathbb{R}^2$ (note that any sufficiently small neighborhood of a point in a C^0 Lorentzian manifold is time orientable; in this case, we can choose the time orientation that agrees with the one we have fixed on Ω), from part 2 above we have that

$$\mathrm{spdiam}\big(I^+(q_n) \cap \Omega\big) \subseteq \mathrm{spdiam}\big(I^+(q_n) \cap I^-(p_n)\big) \xrightarrow{n\to\infty} 0. \tag{20}$$

However, we can explicitly compute the spacelike diameter of the set $I^+(q_n) \cap \Omega$, since this lies entirely inside Ω (i.e. $(\mathcal{N}, h)$): Assuming without loss of generality (by shifting, if necessary, the θ coordinate in the $\mathbb{S}^1$ factor of $\mathbb{N}$) that $q_n = (t_n, \theta_n) \doteq (t_n, 0)$, we can compute from the explicit form of h that the two future directed null curves emanating from q_n (and which mark the boundary of $\mathcal{I}^+(q_n)$) take the form $\gamma_\pm(t) = (t, \theta_\pm(t))$, with

$$\theta'_\pm(t) = \pm t, \quad \theta_\pm(t_n) = 0$$

(so that $\theta_\pm(t) = \pm\frac{1}{2}(t - t_n)^2$). In particular, $I^+(q_n)$ contains the spacelike line segments (for $s \in (0, t_n]$):

$$\gamma_s^{(n)} = \big\{(t,\theta) \in \mathcal{N} :\, t = s,\, -\frac{1}{2}(s-t_n)^2 \leqslant \theta \leqslant \frac{1}{2}(s-t_n)^2\big\}.$$

Note that

$$\lim_{s\to0^+} \ell(\gamma_s^{(n)}) = \lim_{s\to0^+} \int_{-\frac{1}{2}(s-t_n)^2}^{\frac{1}{2}(s-t_n)^2} \frac{1}{\sqrt{s}}\, d\theta = \lim_{s\to0^+} \frac{(s-t_n)^2}{\sqrt{s}} = +\infty$$

and, therefore,

$$\mathrm{spdiam}\big(I^+(q_n)\cap\Omega\big)=+\infty,$$

which is a contradiction in view of (20).